Equivalent Euclidean formulation of special relativity
Application to the lifetime of particles

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Formulation euclidienne équivalente de la relativité restreinte
***Application au temps de vie des particul*es**

Abstract

The classic formulation of special relativity is based on Minkowski's pseudo-Euclidean space-time. However, there is an alternative Euclidean formulation of the theory which gives identical results while considerably simplifying the mathematic tools and graphics necessary for demonstrations. After having established, within the framework of this formulation, the equations of the Lorentz transformation, founders of the theory, it is proposed as an application the measurement of the lifetime of particles, in particular muons.

INTRODUCTION

Shortly after its origin, special relativity was formulated in the framework of a non-Euclidean space (Minkowski pseudo-Euclidean space-time), requiring the use of quadrivectors with pseudonorms that can take negative values and hyperbolic geometry, which makes difficult to read the associated Minkowski diagrams. However, there are several versions of Euclidean special relativity which avoid this type of paradox, but the most developed involve a privileged frame of reference (neo-Lorentzian relativity) [1] and therefore call into question the very foundations of relativity as exposed by Albert Einstein in 1905. These approaches can therefore legitimately leave one skeptical, especially since they lead to results different from those obtained in classical relativity.

The version that we are going to develop here does not call into question the special relativity of Albert Einstein since it leads to exactly the same conclusions as the latter, and as such, it can be perceived as an equivalent formulation to the commonly taught formulation. With the advantage over the latter, however, of using only classical vectors and circular geometry, and, therefore, of establishing relatively familiar diagrams to treat in a particularly visual manner certain classical problems of special relativity, such as for example the lifetime dilation of muons created in the upper atmosphere compared to their proper lifetime.

1. THE BASES OF AN ALTERNATIVE EUCLIDEAN FORMULATION OF SPECIAL RELATIVITY

1.1. From Minkowski space-time to equivalent Euclidean space-time

As in all versions of Euclidean relativity, we will start by modifying the coordinates of Minkowski space-time [2]. In the latter, a particle is located in an inertial reference frame *(R)* by its space coordinate x (to simplify the expressions, we will choose a space-time with one space dimension, instead of the usual three) and its time coordinate t (measured by the clock of an observer at rest in *(R)*), or, better, its coordinate ct, where c is the speed of light in a vacuum, so as to homogenize the coordinates of space and time. The proper time τ of the particle, measured by the clock of the particle, makes it possible to define the space-time interval $s=c\tau$, a quantity invariant by change of inertial reference frame. Figure 1 illustrates the motion, in a one-dimensional Minkowski space-time, of two particles P_0 and P, the first being spatially at rest and the second being moving with a spatial speed v_P, the two particles being initially at the origin O of an inertial reference frame at rest with respect to the observer.

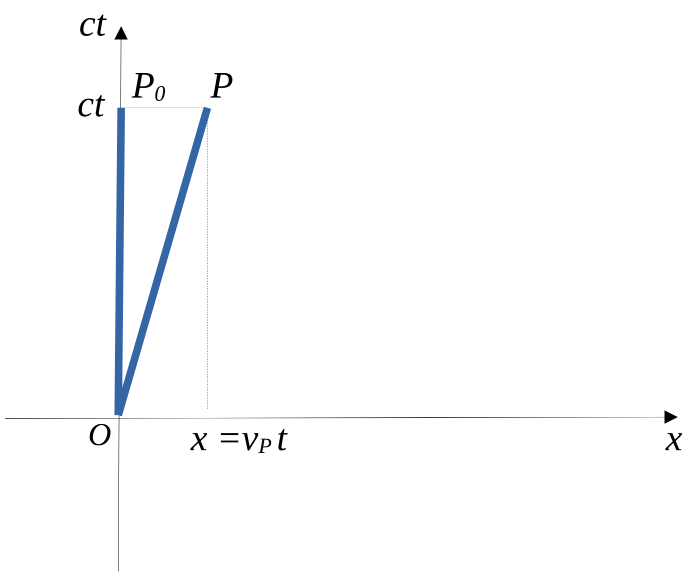

Figure 1 – Representation of the displacement during the duration t of two particles P_0 (spatially at rest) and P (in motion with a constant spatial speed v_P) in a Minkowski space-time with 1 space dimension x (the time coordinate t is replaced by the coordinate ct to homogenize the dimensions of time and space). In this diagram, the segment OP does not represent the space-time interval $s = c\tau$, where τ is the proper time of the particle P, since $s^2 = - x^2 + (ct)^2 \neq x^2 + (ct)^2$.

Minkowski space-time is pseudo-Euclidean. Indeed, in a Euclidean space-time, the square of the space-time interval s of the particle P would be such that $s^2 = x^2 + (ct)^2$, while the particular space-time metric of Minkowski imposes that $s^2 = - x^2 + (ct)^2$. The "–" sign appearing in this metric has a detrimental consequence on the reading of the diagram: the length of the segment OP does not correspond to the value of s.

To move from Minkowski pseudo-Euclidean space-time to a Euclidean space-time, where we find the usual metric of Galilean relativity (with an additional dimension), let's swap the roles of t and x. Thus, in the space-time of our alternative Euclidean formulation, which we call Equivalent Euclidean Formulation of Special Relativity (EEFSR), a particle is identified in an inertial frame of reference (R) by its space coordinate x and its coordinate of proper time τ (or, better, $s=c\tau$, so as to homogenize the dimensions of space and time). Some authors call such a space-time a proper space-time [3], but we continue to call it space-time, for convenience. The time t (or ct) becomes a parameter. However, in the case of a particle P with velocity vector $\mathbf{v_P} = v_P\,\mathbf{e_x}$ in space (with one dimension) emitted in O at time $t = 0$, the parameter ct can be considered equivalent to the coordinate of P along a new ct-axis defined by the unit vector $\mathbf{e} = \mathbf{OP} / ct$, as illustrated in Figure 2 below.

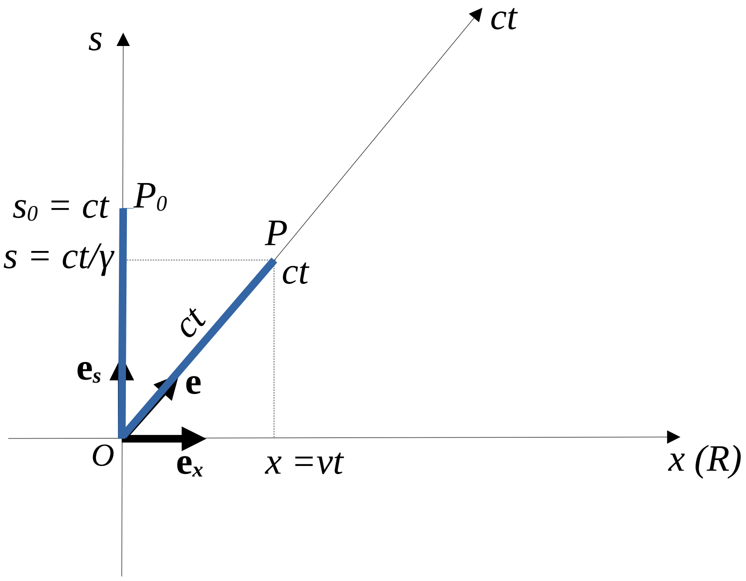

Figure 2 – Euclidean space-time ($\mathbf{e}_x$, $\mathbf{e}_s$ and $\mathbf{e}$ are the unit vectors associated respectively with the *x*-, *s*- and *ct*- axes of an inertial reference frame (*R*)). The particle *P* having a constant spatial speed, the space-time interval *ct* which it travels during the duration *t* corresponds to the coordinate of P along the *ct*-axis. Its coordinate along the *s*-axis is then $s = ct/\gamma$ (cf. Equation (4)). During the same duration *t*, the particle P_0, at rest in (*R*), travels the same space-time interval *ct*, corresponding to the coordinate s_0 of P_0 along the *s*-axis.

In this Euclidean space-time, the position vector of a particle P at time t in the inertial reference frame (R) is written:

$$\mathbf{OP} = x\,\mathbf{e_x} + s\,\mathbf{e_s}. \quad (1)$$

Considering the case of a particle emitted in O at time $t = 0$ moving with a constant spatial speed in the reference frame (R), we can write (see Figure 2):

$$\mathbf{OP} = ct\,\mathbf{e}. \quad (2)$$

By squaring equations (1) and (2), we get:

$$\mathbf{OP}^2 = (ct)^2 = x^2 + s^2$$

that can be written:

$$s^2 = (ct)^2 - x^2. \quad (3)$$

We thus find the expression for the square of the Minkowski space-time interval, an expression which makes it possible to establish, knowing that $x = v\,t$, that:

$$s = ct \,/\, \sqrt{1-v^2/c^2} = ct \,/\, \gamma \qquad (4)$$

(where $\gamma = 1 \,/\, \sqrt{1-v^2/c^2} \geq 1$ is the Lorentz coefficient), expression which also allows us to write that:

$$t = \gamma\,\tau, \qquad (5)$$

relationship which, in special relativity, expresses the “time dilatation” [4-5].

However, in a Euclidean space-time, the space-time interval is not s, but ct (one could set $s = ct$ and define s as the new space-time interval, but, to preserve formal analogy with Minkowski's formulation, it was agreed here to keep $s = c\tau$, where s is now the distance traveled by light during the proper time interval τ). Under these conditions, the diagram in Figure 2 clearly indicates that, for a particle P_0 at rest in (R), whose proper time is equal to t, we have $s_0 = ct > s = ct \,/\, \gamma$ (see equation (4)): the reading of the diagram is therefore consistent with the expected results!

1.2. Representation of a change of reference frame

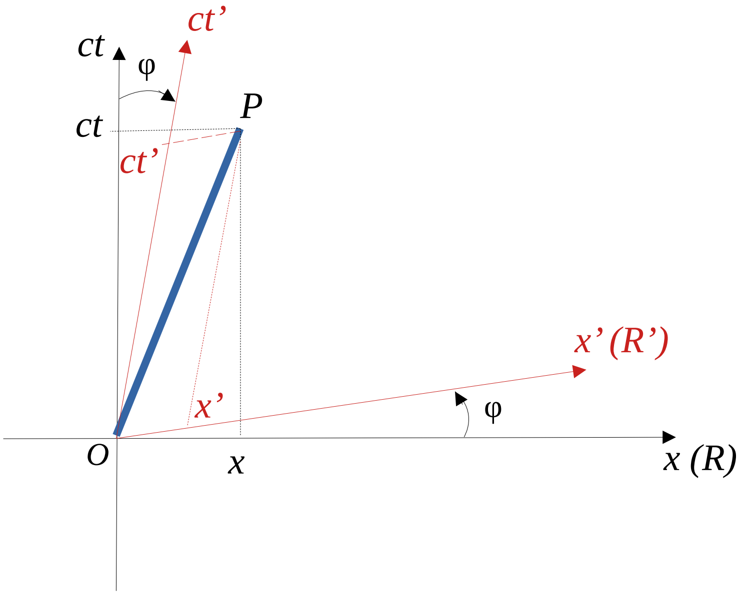

Figure 3 – Representation of a change of reference frame in a Minkowski diagram. The axes of the frame (R) rotate through an angle φ which depends on the speed v of the frame (R’).

How to represent a change of reference frame on a Euclidean space-time diagram? In a Minkowski diagram, the invariance of the space-time interval leads to a *rotation* of the *x*- and *ct*- axes of the reference frame (*R*) (see Figure 3).

Can the same be true in Euclidean space-time? Obviously, no. The invariance of *s* suggests rather that the *s*-axis remains the same for all the inertial reference frames, which imposes a *translation* of the reference frames between them along the *x*-axis. However, seen from the reference frame (*R*), the reference frame (*R'*), moving with a spatial speed *v*, undergoes a length contraction along the *x*-axis, which requires introducing unit vectors "seen from (*R*)", like $(\mathbf{e}_{x'})_R$ and $(\mathbf{e'})_R$ in the diagram Figure 4, or "seen from (*R'*)", like $(\mathbf{e}_x)_{R'}$ and $(\mathbf{e})_{R'}$ in the diagram Figure 5. Indeed, the invariance of the speed of light allows to establish the following relationships (see Appendix 1):

$$(\mathbf{e}_{x'})_R = \mathbf{e}_x / \gamma \tag{6}$$

$$(\mathbf{e}_x)_{R'} = \mathbf{e}_{x'} / \gamma. \tag{7}$$

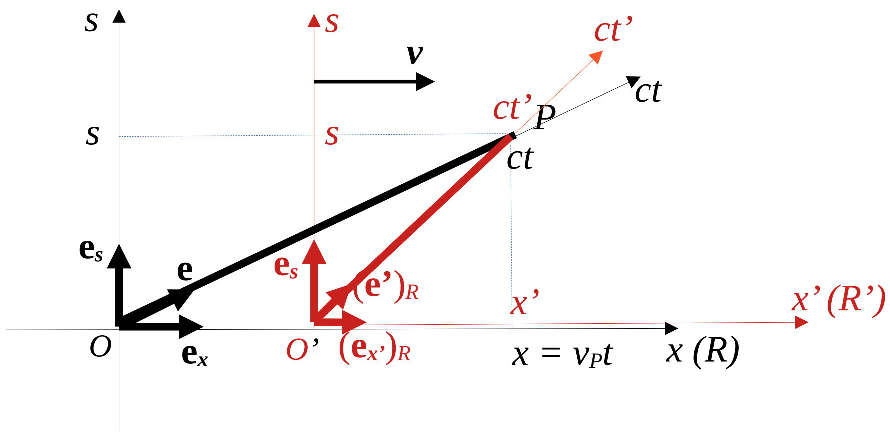

Figure 4 – Euclidean space-time diagram showing a particle *P* and an inertial reference frame (*R'*) as seen from the inertial frame (*R*). The reference frame (*R'*), in translation along the *x*-axis at the spatial speed v, undergoes a length contraction along the *x*-axis such that the unit vector $\mathbf{e}_{x'}$ seen from (*R*) becomes $(\mathbf{e}_{x'})_R = \mathbf{e}_x / \gamma$ (see Appendix 1), which also leads to a contraction of the vector **e'**.

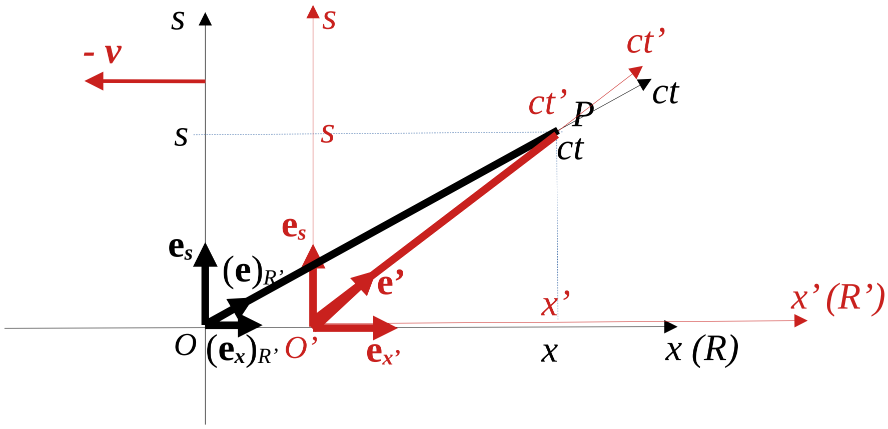

Figure 5 – The same situation as in figure 4, but seen from the inertial reference frame (*R'*). This time, it is the unit vectors $\mathbf{e}_x$ and $\mathbf{e}$ of (*R*) which are contracted, with $(\mathbf{e}_x)_{R'} = \mathbf{e}_{x'} / \gamma$.

Let us now see how it is possible to find the equations of the Lorentz transformation during a change of inertial reference frame in a Euclidean space-time.

2. EEFSR AND LORENTZ TRANSFORMATION

2.1. Transformation of *x* and *x'* coordinates

Consider a particle *P* moving, in a reference frame (*R*), with a spatial speed v_P. Initially, the particle *P* is at the origin *O* of an inertial reference frame (*R*) (see Figure 4). Under these conditions, the position vector of the particle *P* at time *t* is given by Equations (1) and (2), hence:

$$\boldsymbol{OP} = x\,\mathbf{e}_x + s\,\mathbf{e}_s = ct\,\mathbf{e}. \quad (8)$$

In a reference frame (*R'*) in uniform rectilinear motion with respect to (*R*) such that $\mathbf{v}_{R'/R} = v\,\mathbf{e}_x$, whose origin *O'* coincides with *O* at time $t = t' = 0$, the position vector of the particle *P* becomes (see Figure 5):

$$\boldsymbol{O'P} = x'\,\mathbf{e}_{x'} + s\,\mathbf{e}_s = ct'\,\mathbf{e}'. \quad (9)$$

However, as we can see in the diagram Figure 4, seen from (*R*), the position vector of *P* in (*R'*) becomes, taking into account Equation (6):

$$(\boldsymbol{O'P})_R = x'\,(\mathbf{e}_{x'})_R + s\,\mathbf{e}_s = (x'/\gamma)\,\mathbf{e}_x + s\,\mathbf{e}_s. \quad (10)$$

However, the diagram Figure 5 also allows us to write, since $x_{O'} = vt$, that:

$$\boldsymbol{(O'P)}_R = (x - x_{O'})\,\mathbf{e}_x + s\,\mathbf{e}_s = (x - vt)\,\mathbf{e}_x + s\,\mathbf{e}_s. \tag{11}$$

Comparing Equations (10) and (11), it comes:

$$x' = \gamma\,(x - vt). \tag{12}$$

This is the well-known equation of the Lorentz transformation relative to the space coordinate x.

The inverse equation is obtained in the same way, but by reasoning from the diagram Figure 5 and taking into account the Equation (7):

$$\boldsymbol{(OP)}_{R'} = x\,(\mathbf{e}_x)_{R'} + s\,\mathbf{e}_s = (x/\gamma)\,\mathbf{e}_x + s\,\mathbf{e}_s \tag{13}$$

and:

$$\boldsymbol{(OP)}_{R'} = (x' - x'_O)\,\mathbf{e}_x + s\,\mathbf{e}_s = (x' + vt')\,\mathbf{e}_x + s\,\mathbf{e}_s \tag{14}$$

since $x_O' = -vt'$.

By comparing Equations (13) and (14), we find the wanted equation:

$$x = \gamma\,(x' + vt'). \tag{15}$$

2.2. Transformation of coordinates *t* and *t'*

The equations relating to t and t' can be obtained using Equation (3). Indeed, taking into account the invariance of s, we can write that:

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2.$$

By replacing x' by its expression from Equation (12), we obtain:

$$(ct)^2 - x^2 = (ct')^2 - \gamma^2(x - vt)^2.$$

This relationship makes it possible to establish, after development and simplification (see Appendix 2), that:

$$ct' = \gamma\,(ct - vx/c). \tag{16}$$

By replacing x in this last equation by its expression from Equation (15) and rearranging the terms, we obtain the inverse equation:

$$ct = \gamma\,(ct' + vx'/c). \tag{17}$$

2.3. Consequences

The Equations (12), (15), (16) and (17) correspond to the equations of the Lorentz transformation during a change of inertial reference frame. They are the basis of special relativity stated by Albert Einstein. They allow, for example, to find the relativistic composition law for velocities (see Appendix 2):

$$v_p' = \frac{v_P - v}{1 - v_p . v/c^2} \quad . \tag{18}$$

As surprising as it may seem, EEFSR is indeed a rigorous formulation of special relativity in a purely Euclidean framework!

3. APPLICATION TO MEASURING THE LIFETIME OF PARTICLES

3.1. Lifetime dilation of a moving particle

Let us consider, in an inertial reference frame (R), two particles P_0 and P_1 with the same proper lifetime τ_V (measured at the clock of each particle) created simultaneously in O at time $t = 0$. If these two particles are spatially at rest in (R), their decays will obviously be simultaneous[1] at time $t = \tau_V$ (see Figure 6).

What happens now if the particle $P1$ is created, no longer spatially at rest, but with the spatial speed v? As shown in the diagram in Figure 7, when P_0 decays at time $t_0 = \tau_V$, the particle P_1 does not decay, since its proper time is then less than τ_V. Indeed, as this same diagram shows, the decay of P_1 will only take place at time $t_1 = \gamma\,\tau_V > t_0$: for an observer spatially at rest in (R), the decay of the two particles does not is no longer simultaneous, the particle in motion decays after the particle at rest.

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1 We consider here that the particles decay after a time equal to τ_V. As τ_V is in reality an average lifetime, experimental verification cannot be carried out than on a large number of particles [4].

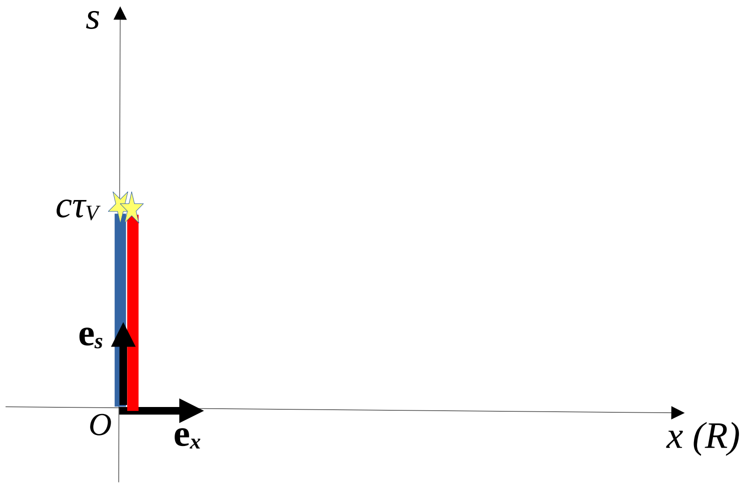

Figure 6 – Simultaneous decays of two particles P_0 (blue path) and P_1 (red path) with the same proper lifetime τ_V at rest in (R).

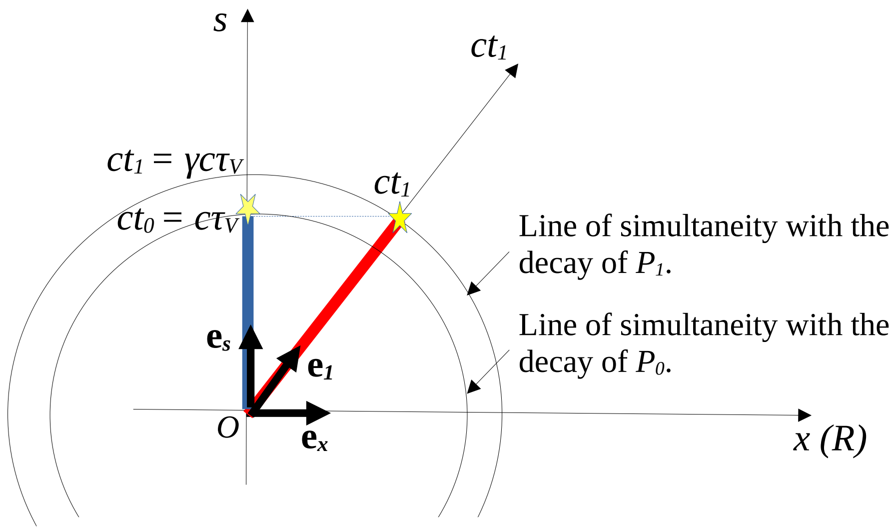

Figure 7 – Decays in (R) of two particles P_0 (at rest, blue path) and P_1 (in motion, red path) with the same proper lifetime τ_V. The two decays are not on the same line of simultaneity: in (R), the moving particle decays after the resting particle P_0.

Thus, the particle P_1, instead of traveling the distance $d_0 = v\tau_V$ which we would expect if relativistic effects did not exist, travels, taking into account special relativity, the distance (see Figure 8):

$$d = x_1 = v\tau_V = v\,\gamma\,\tau_V = \gamma\, d_0 > d_0. \tag{19}$$

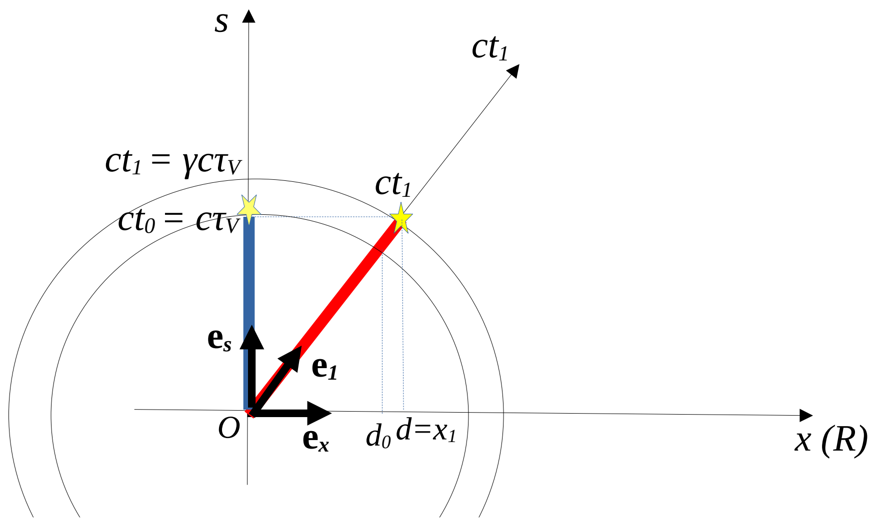

Figure 8 – The distance $d = x_1 = vt_1 = \gamma v\tau_V$ traveled in (R) by the particle P_1 during its lifetime t_1 is greater than the distance $d_0 = v.\tau_V$ that the particle would have traveled if the relativistic effects did not exist.

This result is confirmed by the study of the average distance traveled in the atmosphere by muons, particles created at several kilometers of altitude with a speed close to that of light. Indeed, the average proper lifetime of muons is 2.2 μs, which would allow them, if relativistic effects did not exist, to travel on average around 650 m. However, observations show that a number of muons pass through the atmosphere until reaching the ground, which confirms that their average lifetime is much greater than 2.2 μs, as predicted by special relativity. For example, at a speed of 0.995c (i.e. for a Lorentz factor $\gamma = 10$), we calculate an average lifetime of 22 μs and an average distance traveled of approximately 6.5 km, which is confirmed by experiments [4].

3.2. Moving particle point of view

What would we observe in a reference frame (R') in motion with a spatial speed $\mathbf{v}_{R'/R} = v\ \mathbf{e}_x$ with respect to (R), therefore in a reference frame in which the particle P_1 would be at rest and the particle P_0 moving with a speed $\mathbf{v}_0' = -v\ \mathbf{e}_x$? The diagram in Figure 9 represents this situation. We can see that, if the particle P_1 (red path) decays, as it should, at the instant $t_1' = \tau_V$, the particle P_0 decays at the instant $t_0' = \gamma\ \tau_V > t_1'$, therefore after the particle P_1. This, of course, is not surprising, since the situation in (R') is the reciprocal of the situation in (R), with, this time, P_1 at rest and P_0 in motion (note that the lines of simultaneity with a given event do not coincide in (R) and (R'):

as predicted by special relativity, two simultaneous events in (R) are no longer simultaneous in (R')).

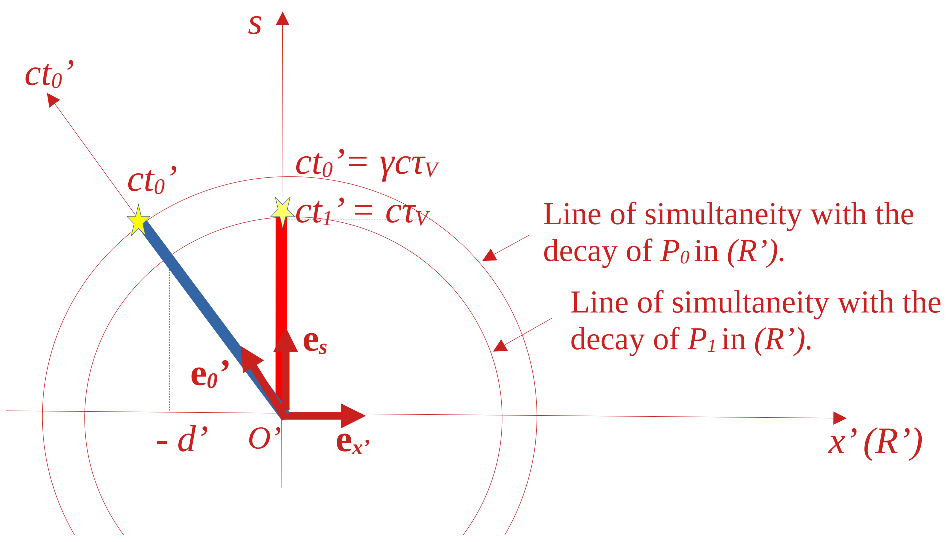

Figure 9 – In the reference frame (R') attached to $P1$, $P1$ decays at the date $t_1' = \tau_V$ (red path) while P_0 decays at the later date $t_0' = \gamma\tau_V$ (blue path). Furthermore, the distance traveled by a point at rest in (R), such as for example P_0, will be equal to $d' = v\tau_V = d / \gamma$.

3.3. Relativistic length contraction

Finally, it is interesting to note that, when P_1 decays, the distance traveled in (R') by the particle P_0 (and therefore by any point at rest in (R)) is equal to (see Figure 9):

$$d' = v\,\tau_V = d / \gamma < d. \tag{20}$$

We conclude that an observer attached to a muon with a proper lifetime $\tau_V = 2.2$ ns moving at a speed of $0.995c$ will consider having traveled in the Earth's atmosphere the distance $d' = 650$ m, and not the distance $d = 6.5$ km measured by a terrestrial observer, between the creation of the muon and its decay. We find length contraction in the direction of displacement introduced in subparagraph 1.2., which can be written here, taking into account Equation (7):

$$d'\,\mathbf{e}_{x'} = d\,(\mathbf{e}_x)_{R'} = d\,\mathbf{e}_{x'} / \gamma. \tag{21}$$

We can illustrate Equation (21) by the diagram in Figure 10 below:

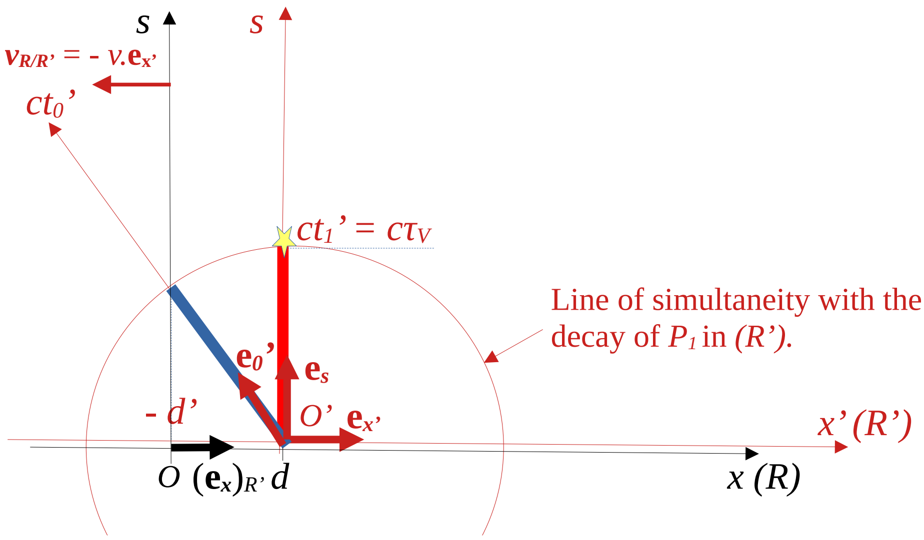

Figure 10 – When P_1 (red path) decays, its spatial displacement in (*R*) seen from (*R'*) can be written $(\boldsymbol{OO'})_{R'} = d.(\mathbf{e}_x)_R$. Simultaneously, the particle P_0 (blue path) moved in (*R'*) according to $(\boldsymbol{O'O})_{R'} = - d'.\mathbf{e}_{x'}$. It follows that $(\boldsymbol{OO'})_{R'} = d'.\mathbf{e}_{x'} = d.(\mathbf{e}_x)_{R'} = d.\mathbf{e}_{x'} / \gamma$.

This example shows how the EEFSR, thanks to the vector $(\mathbf{e}_x)_{R'}$, unit vector $\mathbf{e}_x$ "seen from (*R'*)", makes it possible to visualize the exclusively relativistic effect of length contraction of a moving reference frame [5-6].

CONCLUSION

EEFSR is a reformulation of special relativity in a purely Euclidean geometric framework. As such, its predictive power is the same as that of special relativity (see the equations of the Lorentz transformation). However, its geometric framework, similar to the geometric framework of Galilean relativity, but with an additional dimension, tends to present EEFSR as a "natural" formulation of special relativity, even if, historically, it is the pseudo-Euclidean formulation of Minkowski who prevailed.

But the major educational interest of this Euclidean formulation is that it allows the quantities *ct*, *x* and *s* to be represented on the same diagram (which is not possible, for example, with a Minkowski diagram), which can help to visualization of relativistic effects not always easy to understand by people unfamiliar with

Minkowski's formalism, such as the time dilation or the length contractions which appear during the study of very fast particles like muons.

Why then not imagine the EEFSR and its diagrams as a bridge between Galilean relativity and special relativity in its Minkowski formulation, whose pseudo-Euclidean metric, although less easy to master than the Euclidean metric, is nevertheless necessary, currently, to the expression of modern theories of physics[2]?

ACKNOWLEDGMENT

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REFERENCES

[1] J. M. C. Montanus, "Proper Time Physics", *Hadronic Journal*, n° 22, 1999.

[2] Euclidean Relativity : https://www.euclideanrelativity.com

[3] R.G. Newburgh and Jr T.E. Phipps, "A Space-Proper Time Formulation of Relativistic Geometry", *Physical Sciences Research Paper*, November 1969.

[4] P. Magnien, *Désintégration du muon : une horloge relativiste* https://vaubsciences-56.webself.net/file/si76624/download/DESINTEGRATION%20DU%20MUON_UNE%20HORLOGE%20RELATIVISTE-fi2454489.pdf

[5] CLEA file on special relativity: http://clea-astro.eu/lunap/Relativite/relativite-restreinte-principes-et-applications/ See in particular the article entitled «De la dilatation des temps à la contraction des longueurs», by Pierre Magnien.

[6] J.-M. Vigoureux, *L'Univers en perspective*, Ellipse, 2006.

2 About this, see the excellent videos by Alessandro Roussel on the ScienceClic site, in particular those concerning general relativity.

APPENDIX 1

Unit space vector of a reference frame in motion seen from a reference frame at rest

To determine the relationship between the unit vector $\mathbf{e}_x$ and the unit vector $(\mathbf{e}_{x'})_R$, which is the unit vector $\mathbf{e}_{x'}$ "seen from (R)", we will imagine the situation where photons are emitted at time $t = t' = 0$ by a source S and are received by the observers Ω and Ω' attached respectively to the reference frames (R) and (R'), (R) being considered at rest and (R') moving at speed v with respect to (R). The diagram in Figure A1.1 below represents this situation.

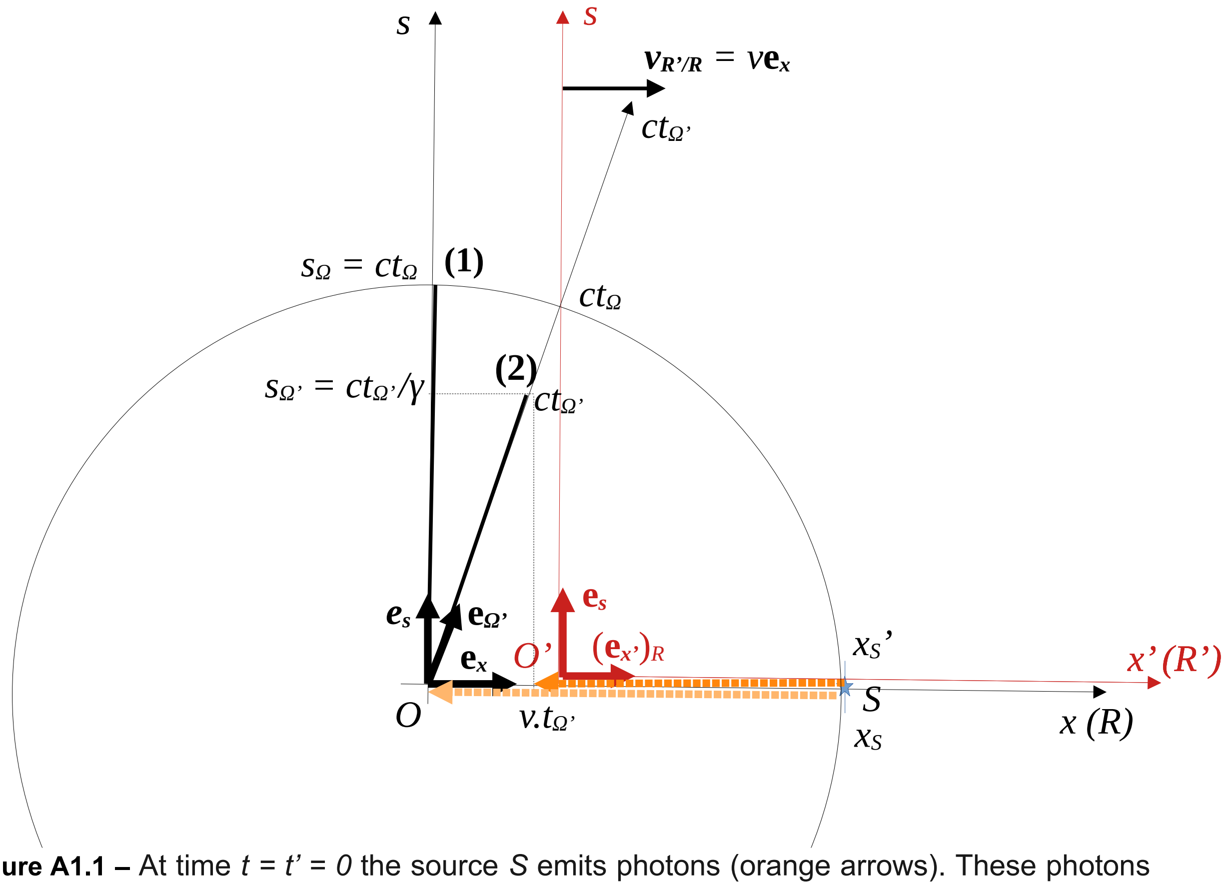

Figure A1.1 – At time $t = t' = 0$ the source S emits photons (orange arrows). These photons reach the observer Ω (light orange arrow), spatially at rest in (R), at time $t_\Omega = x_S / c$ (Ω is then in position (1)). They reach the observer Ω' (dark orange arrow), stationary in the reference frame (R') which goes at speed v in (R), at time $t_{\Omega'}$ (Ω' is then in position (2), with the coordinate $s_{\Omega'} = ct_{\Omega'}/\gamma$ along the s-axis). We show that the instant $t_{\Omega'}$ is such that the abscissa x_S' of the source S measured in (R') verify the relation $x_S'.(\mathbf{e}_{x'})_R = (ct_{\Omega'}/\gamma^2).\mathbf{e}_x$, where $(\mathbf{e}_{x'})_R$ is the unit vector $\mathbf{e}_{x'}$ carried by (R') (which, at rest, is equal to $\mathbf{e}_x$) « as seen from the reference frame (R) ». *A priori* $(\mathbf{e}_{x'})_R \neq \mathbf{e}_x$.

As photons move through space at speed c, they have no proper time. Observer Ω, attached to the reference frame (R), will therefore receive the photons at time $t_\Omega = x_S/c$ (position (1) on the diagram). At what time $t_{\Omega'}$ will observer Ω' receive the photons (position (2) on the diagram)? We read on the diagram that it is the instant $t_{\Omega'}$ such that:

$$x_S = ct_\Omega = ct_{\Omega'} + v.t_{\Omega'},$$

$ct_{\Omega'}$ being the distance traveled by the photons, in (R), between the source S and their reception by Ω' (dark orange arrow in figure A1.1). We deduce that:

$$ct_\Omega = ct_{\Omega'}(1+v/c).$$

However, still according to the diagram in Figure A1.1:

$$x_S'(\mathbf{e}_{x'})_R = (x_S - x_{O'})\,\mathbf{e}_x = (ct_\Omega - vt_\Omega)\,\mathbf{e}_x = ct_\Omega(1\text{-}v/c)\,\mathbf{e}_x$$

since $x_S = ct_\Omega$ et $x_{O'} = v\,t_\Omega$.

We therefore deduce that:

$$x_S'(\mathbf{e}_{x'})_R = ct_{\Omega'}(1+v/c)\,(1\text{-}v/c)\,\mathbf{e}_x = ct_{\Omega'}(1\text{-}v^2/c^2)\,\mathbf{e}_x = (ct_{\Omega'}/\gamma^2)\,\mathbf{e}_x. \qquad (22)$$

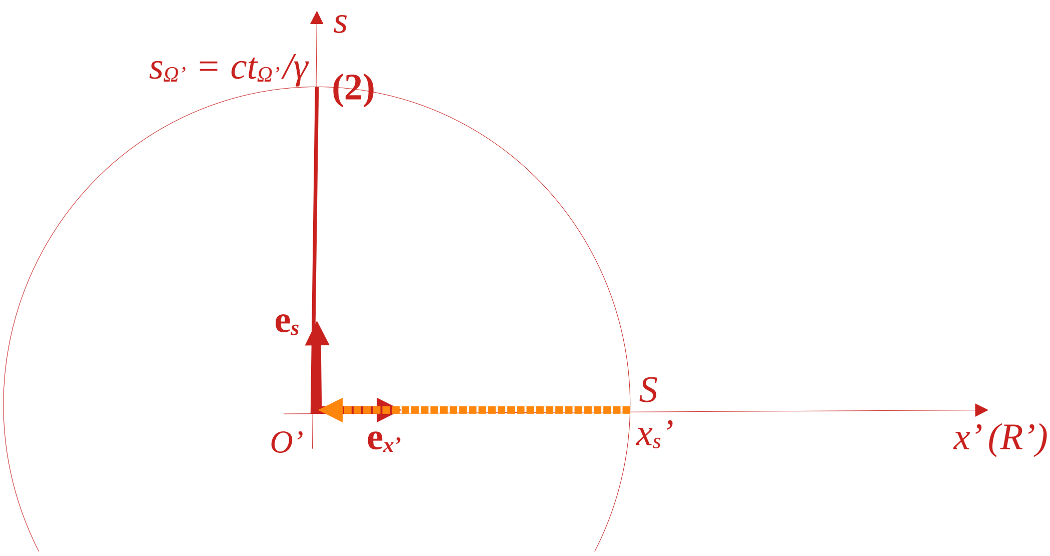

Figure A1.2 – In the reference frame (R') the source S is at the abscissa x_S'. The photons emitted by S (dark orange arrow) reach the observer Ω' at rest in (R') at time $t_{\Omega'}$ (Ω' is then in position (2), with the invariant coordinate $s_{\Omega'} = ct_{\Omega'}/\gamma$ along the s-axis). The abscissa x_S' therefore verifies the relationship $x_S' = s_{\Omega'} = ct_{\Omega'}/\gamma$. As $x_S'.(\mathbf{e}_{x'})_R = (ct_{\Omega'}/\gamma^2).\mathbf{e}_x$ (see figure A1.1), we deduce that $(\mathbf{e}_{x'})_R = \mathbf{e}_x/\gamma$.

Let us now observe the situation, represented in the diagram in Figure A1.2, in the reference frame (*R'*).
It immediately appears that:

$$x_S' = s_{\Omega'} = ct_{\Omega'} / \gamma. \tag{23}$$

Equations (22) and (23) therefore imply that:

$$(ct_{\Omega'} / \gamma)\, (\mathbf{e}_{x'})_R = (ct_{\Omega'}/\gamma^2)\, \mathbf{e}_x$$

from which we deduce that:

$$(\mathbf{e}_{x'})_R = \mathbf{e}_x / \gamma.$$

The relationship $(\mathbf{e}_x)_{R'} = \mathbf{e}_{x'}/\gamma$ is obtained in the same way, but by first starting from the situation seen from (*R'*), then by comparing it to the situation seen from (*R*).

APPENDIX 2
Transformation law for time *t*
Relativistic composition law for velocities

TRANSFORMATION LAW FOR TIME *t*

The invariance of s allows us to write:

$$s^2 = (ct)^2 - x^2 = (ct')^2 - x'^2 \text{ avec } x' = \gamma(x - vt)$$

hence:

$$(ct)^2 - x^2 = (ct')^2 - \gamma^2(x - vt)^2$$

By expanding this expression we obtain:

$$(ct)^2 - x^2 = (ct')^2 - \gamma^2x^2 - \gamma^2v^2t^2 + 2\gamma^2xvt = (ct')^2 - \gamma^2x^2 - \gamma^2\beta^2(ct)^2 + 2\gamma^2x\beta ct, \text{ with } \beta = v/c,$$

hence :

$$(ct')^2 = (ct)^2 - x^2 + \gamma^2x^2 + \gamma^2\beta^2(ct)^2 - 2\gamma^2x\beta ct = (ct)^2(1 + \gamma^2\beta^2) - x^2(1 - \gamma^2) - 2\gamma^2x\beta ct.$$

Now:

$$1 + \gamma^2\beta^2 = \gamma^2 \text{ et } 1 - \gamma^2 = -\gamma^2\beta^2$$

so:

$$(ct')^2 = \gamma^2(ct)^2 + \gamma^2\beta^2x^2 - 2\gamma^2x\beta ct = \gamma^2((ct)^2 + \beta^2x^2 - 2x\beta ct) = \gamma^2(ct - \beta x)^2.$$

We deduce that :

$$ct' = \gamma(ct - vx/c).$$

RELATIVISTIC COMPOSITION LAW FOR VELOCITIES

The equations of the Lorentz transformation make it possible to establish the composition law for velocities in special relativity. Indeed, by differentiating $x' = \gamma (x - vt)$ we obtain:

$$dx' = \gamma (dx - vdt) \qquad (24)$$

and by differentiating $ct' = \gamma (ct - vx/c)$ we obtain:

$$cdt' = \gamma\,(cdt - vdx/c). \qquad (25)$$

By dividing Equation (24) by Equation (25), we obtain:

$$dx'/dt' = (dx - vdt) / (dt - vdx/c^2) = (dx/dt - v) / (1 - v(dx/dt)/c^2).$$

By setting $v_P' = dx'/dt'$ and $v_p = dx/dt$, it comes:

$$v_P' = \frac{v_P - v}{1 - v_p . v/c^2} \ .$$